

LEVEL J

J 1-10 : Expansion of Polynomial Products

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$$

$$(a + b)(a - b) = a^2 - b^2$$

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

$$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$$

J 11-60 : Factorisation

$$acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$$

$$2x^2 + 5x - 12 = (2x - 3)(x + 4)$$

$$\begin{array}{cc} 2 & -3 \\ 1 & 4 \end{array}$$

$$a^2 \pm 2ab + b^2 = (a \pm b)^2$$

$$a^2 - b^2 = (a + b)(a - b)$$

“Taking out” the common factor, e.g.

$$a(x + y) + x + y = (x + y)(a + 1)$$

$$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$$

Extra notes:

$$(x - y)^2 = (y - x)^2 \quad (x - y)^4 = (y - x)^4$$

$$(x - y) = -(y - x) \quad (x - y)^3 = -(y - x)^3$$

For complicated quadratic expressions with multiple variables (e.g. x, y, z),

- 1) Arrange terms in descending powers of x
- 2) Factorise terms which are independent of x
- 3) Factorise in quadratic terms, e.g. $(x + \blacksquare)(x + \triangle)$
- 4) Simplify content in the brackets

$$\begin{aligned} & 2a^2 + 2b^2 + c^2 + 4ab + 3ac + 3bc \\ &= 2a^2 + (4b + 3c)a + (2b^2 + 3bc + c^2) \\ &= 2a^2 + (4b + 3c)a + (2b + c)(b + c) \\ &= [2a + (2b + c)][a + (b + c)] \\ &= (2a + 2b + c)(a + b + c) \end{aligned}$$

$$\left[\begin{array}{cc} 2 & 2b+c \rightarrow 2b+c \\ 1 & b+c \rightarrow 2b+2c \\ & 4b+3c \end{array} \right]$$

Useful trick 1:

$$\begin{aligned} & x^4 - 6x^2 + 1 \\ &= x^4 - 2x^2 + 1 - 4x^2 \\ &= (x^2 - 1)^2 - (2x)^2 \\ &= (x^2 - 1 + 2x)(x^2 - 1 - 2x) \\ &= (x^2 + 2x - 1)(x^2 - 2x - 1) \end{aligned}$$

Useful trick 2:

$$\begin{aligned} & (x^2 + x - 5)(x^2 + 2x - 5) - 12x^2 \\ &= (A + x)(A + 2x) - 12x^2 \quad [\text{where } A = x^2 - 5] \\ &= A^2 + 3xA - 10x^2 \\ &= (A + 5x)(A - 2x) \\ &= (x^2 + 5x - 5)(x^2 - 2x - 5) \end{aligned}$$

Useful trick 3:

Arrange terms in descending powers of the variable which has the highest power of 2 or 1.

$$\begin{aligned} & x^3 + (2a + 1)x^2 + (a^2 + 2a - 1)x + (a^2 - 1) \\ &= (x + 1)a^2 + 2x(x + 1)a + x^3 + x^2 - x - 1 \end{aligned}$$

$$\begin{aligned} & ax^2 - a^3 - a^2b + ab^2 + b^3 - bx^2 \\ &= (a - b)x^2 - (a^3 - b^3) - ab(a - b) \end{aligned}$$

J 61-70 : Fractional Expressions

To reduce fractional expressions, first **factorise** the numerator and denominator

$$\frac{-x^2 + 5x - 6}{x^2 - 7x + 12} = \frac{-(x - 3)(x - 2)}{(x - 4)(x - 3)} = -\frac{x - 2}{x - 4}$$

To combine two or more fractional expressions (adding/subtracting), first seek the LCM

$$\frac{x - 5}{x + 5} + \frac{x + 5}{x - 5} = \frac{(x - 5)^2 + (x + 5)^2}{(x + 5)(x - 5)} = \frac{2(x^2 + 25)}{(x + 5)(x - 5)}$$

Simplifying complex fractions by eliminating their denominators

$$\frac{1 + \frac{1}{a}}{1 - \frac{1}{a}} = \frac{\left(1 + \frac{1}{a}\right)a}{\left(1 - \frac{1}{a}\right)a} = \frac{a + 1}{a - 1}$$

J 71-90 : Irrational Numbers

Rationalise the denominators

$$\frac{\sqrt{3}}{\sqrt{5}} = \frac{\sqrt{3} \times \sqrt{5}}{\sqrt{5} \times \sqrt{5}} = \frac{\sqrt{15}}{5}$$

$$\frac{1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{(\sqrt{3}+1)(\sqrt{3}-1)} = \frac{\sqrt{3}-1}{2}$$

Using the fact that $(\sqrt{a} \pm \sqrt{b})^2 = a + b \pm 2\sqrt{ab}$

$$\Rightarrow \sqrt{a+b \pm 2\sqrt{ab}} = \sqrt{a} \pm \sqrt{b} \quad \text{where } a > b$$

$$\begin{array}{cc} \sqrt{7+2\sqrt{10}} = \sqrt{5} + \sqrt{2} & \sqrt{8-2\sqrt{15}} = \sqrt{5} - \sqrt{3} \\ \uparrow \quad \uparrow & \uparrow \quad \uparrow \\ 5+2 \quad 5 \times 2 & 5+3 \quad 5 \times 3 \end{array}$$

Note: The above formula is only possible when there is a “2” next to the square root.

e.g.

$$\begin{aligned} \sqrt{2+\sqrt{3}} &= \sqrt{\frac{4+2\sqrt{3}}{2}} = \frac{\sqrt{4+2\sqrt{3}}}{\sqrt{2}} \\ &= \frac{\sqrt{3}+1}{\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{2} \end{aligned}$$

The concept of principal square root is analogous to that of absolute value

When $a \geq 0$, (a is 0 or a positive number), $\sqrt{a^2} = a$ When $a < 0$, (a is a negative number), $\sqrt{a^2} = -a$ When $a \geq b$, $\sqrt{(a-b)^2} = a-b$ When $a < b$, $\sqrt{(a-b)^2} = -(a-b)$ **J 91-110 : Quadratic Equations**

Technique 1: Factorisation

$$6x^2 - x = 12$$

$$[\text{Sol}] \quad 6x^2 - x - 12 = 0$$

$$(3x+4)(2x-3) = 0$$

$$x = -\frac{4}{3}, \frac{3}{2}$$

Technique 2: Solve as “linear equation”

$$x^2 - 18 = 0$$

$$x^2 = 18$$

$$x = \pm 3\sqrt{2}$$

Technique 3: Completing the square

$$x^2 - 5x - 7 = 0$$

$$[\text{Sol}] \quad x^2 - 5x = 7$$

$$x^2 - 5x + \left(\frac{5}{2}\right)^2 = 7 + \left(\frac{5}{2}\right)^2$$

$$\left(x - \frac{5}{2}\right)^2 = \frac{53}{4}$$

$$x - \frac{5}{2} = \pm \frac{\sqrt{53}}{2}$$

$$x = \frac{5 \pm \sqrt{53}}{2}$$

Technique 4: Quadratic Formula

Quadratic Formula IWhen $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula IIWhen $ax^2 + 2b'x + c = 0$,

$$x = \frac{-b' \pm \sqrt{b'^2 - ac}}{a}$$

J 111-120 : Complex Numbers

$$\sqrt{-1} = i \quad \text{e.g. } \sqrt{-5} = \sqrt{5}i$$

$$i^2 = -1$$

Imaginary numbers/complex numbers work the same way as algebra/real numbers.

$$\sqrt{-4} \times \sqrt{-25} = 2i \times 5i = 10i^2 = -10$$

$$(2+i)(3+2i) = 6 + 7i + 2i^2 = 4 + 7i$$

$$i^3 = i^2 \cdot i = (-1) \cdot i = -i$$

$$i^4 = (i^2)^2 = (-1)^2 = 1$$

Avoid having complex numbers in the denominator

$$\begin{aligned} \frac{3-i}{3+2i} &= \frac{(3-i)(3-2i)}{(3+2i)(3-2i)} \\ &= \frac{9-9i+2i^2}{9+4} \\ &= \frac{7-9i}{13} \end{aligned}$$

J 121-130 : Discriminant

The discriminant tells us what type of solutions/roots a quadratic equation has

For $ax^2 + bx + c = 0$,
the discriminant is

$$D = b^2 - 4ac$$

For $ax^2 + 2b'x + c = 0$,
use

$$\frac{D}{4} = b'^2 - ac$$

$D > 0 \Leftrightarrow$ There are 2 different real solutions.

$D = 0 \Leftrightarrow$ There is 1 repeated real solution.

$D < 0 \Leftrightarrow$ There are 2 different complex (conjugate) solutions.

J 131-140 : Root-Coefficient Relationships

Given $ax^2 + bx + c = 0$ ($a \neq 0$),
if the roots are α and β ,

$$\alpha + \beta = -\frac{b}{a} \quad \alpha\beta = \frac{c}{a}$$

Extra Notes: Given $\alpha + \beta$ and $\alpha\beta$, one might appreciate the following identities.

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$$

Alternatively, one can use

$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$$

J 141-150 : Simultaneous Equations

Useful trick 1: Substitution

$$\begin{cases} y = 2x - 1 & \dots\dots(1) \\ x^2 + y^2 = 29 & \dots\dots(2) \end{cases}$$

[Sol] Substituting (1) into (2),
$$x^2 + (2x - 1)^2 = 29$$

Useful trick 2: Eliminate “troublesome” terms

$$\begin{cases} x^2 + y^2 - 2x + 2y = 7 & \dots\dots(1) \\ x^2 + y^2 + 4x - 4y = 1 & \dots\dots(2) \end{cases}$$

[Sol] From (1) - (2),

$$-6x + 6y = 6$$

$$\text{Therefore, } x = y - 1 \quad \dots\dots(3)$$

Substituting (3) into (1)

Useful trick 3: Obtain relationships between x and y from equation (1)

$$\begin{cases} (x - y)(x + 2y) = 0 & \dots\dots(1) \\ x^2 - xy + 2y^2 = 16 & \dots\dots(2) \end{cases}$$

[Sol] From (1), $x = y$ or $x = -2y$

$$\begin{cases} x = y & \dots\dots(3) \\ x^2 - xy + 2y^2 = 16 & \dots\dots(2) \end{cases} \quad \begin{cases} x = -2y & \dots\dots(3') \\ x^2 - xy + 2y^2 = 16 & \dots\dots(2) \end{cases}$$

Useful trick 4: Apply root-coefficient relationships

$$\begin{cases} x + y = 9 & \dots\dots(1) \\ xy = 7 & \dots\dots(2) \end{cases}$$

[Sol] x and y are the roots of

$$t^2 - 9t + 7 = 0$$

$$\text{Therefore, } t = \frac{9 \pm \sqrt{53}}{2}$$

Useful trick 5: Use identities for $x + y$ and xy

$$\begin{cases} xy = 3 & \dots\dots(1) \\ x^2 + y^2 = 10 & \dots\dots(2) \end{cases}$$

[Sol] From (2),

$$(x + y)^2 - 2xy = 10 \quad \dots\dots(3)$$

J 151-160 : Polynomial Division

Algorithm similar to long division for numbers.

Leave a space for terms with coefficient equal zero.

Dividend = Divisor \times Quotient + Remainder

$$\begin{array}{l} \text{Dividend} \quad \quad \text{Divisor} \\ (x^3 - x^2 - 1) \div (x - 2) \\ = x^2 + x + 2 \quad \text{Remainder } 3 \\ \text{Quotient} \end{array}$$

This relationship may be written as

$$x^3 - x^2 - 1 = (x - 2)(x^2 + x + 2) + 3.$$

$$\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}$$

J 161-170 : Remainder Theorem

$P(x)$ is a polynomial,

The Remainder Theorem

When $P(x)$ is divided by $x - a$, the remainder is $P(a)$.

The Remainder Theorem can only be used when dividing by a linear expression, i.e. $ax + b$

Important Note:

Dividing by a **first-degree** expression (a linear), the *remainder* must be of degree **zero** (constant).

Dividing by a **second-degree** expression (a quadratic), the *remainder* must be of degree **one or less** ($ax + b$).

Dividing by a **third-degree** expression (a cubic), the *remainder* must be of degree **two or less** ($ax^2 + bx + c$).

...

Dividing by an n th-degree expression, the remainder must be of degree $(n - 1)$ or less.

J 171-180 : Factor Theorem

$P(x)$ is a polynomial,

The Factor Theorem

$P(x)$ has a factor $(x - a) \Leftrightarrow P(a) = 0$
(\Leftrightarrow means "if and only if")

Useful tricks when guessing the value of a :

List out all factors of the term with the highest power (e.g. x^3) and the constant term.

Try out different combinations until you obtain $P(a) = 0$. In the example below, one might need to try $\pm 1, \pm 2, \pm 5, \pm 10, \pm \frac{1}{2}, \pm \frac{5}{2}$.

$$P(x) = 2x^3 - 3x^2 - x - 10$$

$\begin{array}{c} \swarrow \quad \searrow \\ 1 \quad 2 \end{array} \quad \begin{array}{c} \swarrow \quad \searrow \quad \searrow \\ 1 \quad 2 \quad 5 \end{array} \quad 10$

This may be factorised as $(2x - \square)(x^2 + \triangle x + \square)$ or $(x - \square)(2x^2 + \triangle x + \square)$

$$P\left(\frac{5}{2}\right) = 2 \times \left(\frac{5}{2}\right)^3 - 3 \times \left(\frac{5}{2}\right)^2 - \frac{5}{2} - 10 = 0$$

Try $\pm \frac{1}{2}, \pm \frac{5}{2}$.

Therefore, $(2x - 5)$ is a factor of $P(x)$

J 181-200 : Proof of Identities and Inequalities

If an identity is held true for all values of x , we can:

Compare coefficients

$$(ax + b)(x + 1) = 3x^2 + 5x + 2$$

$$[\text{Sol}] ax^2 + (a + b)x + b = 3x^2 + 5x + 2$$

$$\begin{cases} a = 3 & \dots \textcircled{1} \\ a + b = 5 & \dots \textcircled{2} \\ b = 2 & \dots \textcircled{3} \end{cases}$$

Substitute appropriate values

$$x^2 = a(x - 1)(x - 2) + b(x - 1) + c$$

$$[\text{Sol}] \text{ When } x = 1, \quad 1 = c \quad \dots \textcircled{1}$$

$$\text{When } x = 2, \quad 4 = b + c \quad \dots \textcircled{2}$$

$$\text{When } x = 3, \quad 9 = 2a + 2b + c \quad \dots \textcircled{3}$$

There are 2 methods to prove an **identity**. Example: Given $x + y = 1$, show that $x^2 + y = y^2 + x$.

Method 1: $y = 1 - x$

$$LHS = x^2 + (1 - x) = x^2 - x + 1$$

$$RHS = (1 - x)^2 + x = x^2 - x + 1$$

$$\therefore LHS = RHS$$

Method 2:

$$LHS - RHS = x^2 + y - y^2 - x$$

$$= (x - y)(x + y) - (x - y)$$

$$= (x - y)(x + y - 1) = 0$$

$$\therefore LHS = RHS$$

To prove an **inequality**, we can:

Complete the square (\because a square is non-negative)

Prove $x^2 > x - 1$.

$$[\text{Sol}] (LHS) - (RHS) = x^2 - x + 1$$

$$= \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$\text{Since } \left(x - \frac{1}{2}\right)^2 \geq 0, \quad \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} > 0$$

$$\text{Therefore, } x^2 > x - 1$$

Use the arithmetic mean-geometric mean inequality

$$\text{When } a > 0 \text{ and } b > 0, \text{ then } \frac{a+b}{2} \geq \sqrt{ab}$$

$$LHS = RHS \text{ when } a = b.$$